Why does an apple fall? Introduction to Einsteinian gravity for high schools.

David Blair[1], Tejinder Kaur[1], Marjan Zadnik[1] ,Warren Stannard[1]

[1]The University of Western Australia, Crawley, WA 6009

Abstract

The falling apple epitomises Newton's gravity: a force exerted by planet Earth. Although this concept of gravity was overthrown by Einstein's theory of General Relativity, it is rarely taught in high school. Here we present an approach to Einsteinian gravity: the force required to prevent free fall. Our approach, based on spacetime diagrams, warped time and the principle of maximal aging, has been tested successfully with students ranging from 11-17 years of age.

I. Introduction

The equivalence principle is the observation that the acceleration of freely falling bodies has no dependence on mass or composition. This led Einstein to his radical re-formulation of the concept of gravity: gravity is not a force field that emanates out of planets and other massive bodies, but rather a manifestation of curved spacetime. While the beauty of this concept is apparent to all who study it, it has normally been avoided in school, where the old Newtonian concepts are normally taught.

Newton said that space is flat and time is absolute. According to Newton, space and time are independent. The geometry of space is Euclidean. The presence of matter does not affect space. The gravitational force field is responsible for apples falling and for keeping the Earth in motion around the Sun.

- Einstein reversed all of the above concepts. He said that space is non-Euclidean, that time is not absolute, and that spacetime is curved by the presence of matter. *Matter tells spacetime how to curve, and spacetime tells matter how to move.* (1)

The basic concepts of general relativity are normally introduced by illustrating Wheeler's phrase with a rubber diaphragm such as a stretched lycra sheet on which suitable balls such as golf balls roll in orbits. As we discuss below, this is a reasonable analogy for planetary orbits, and also allows curved space effects such as the deflection of light beams to be understood.

The rubber sheet analogy emphasises the *spatial* curvature introduced by matter – and helps students develop an intuitive understanding of the geodetic effects that arise because of the breakdown of Euclidean geometry. Specifically Kepler's second law –the law of equal areas – states that planets sweep out equal areas in equal times. The third law relates the period of a planetary orbit to its distance from the sun. In non-Euclidean space orbits must precess because these two formulations of Keplerian dynamics are non-commensurate in non-Euclidean space. This is well illustrated on a rubber sheet in which the area of a circle clearly can increase independently of its perimeter – the area to circumference ratio is no longer equal to radius/2 as it is in Euclidean geometry.

While orbital dynamics can be simply illustrated using a rubber sheet, the fall of an apple is less clear, because gravity on Earth is almost entirely explained by the warping of time. Here we present our approach, which uses four steps. First we make a systematic introduction to spacetime, so as to be able to consider the length of spacetime trajectories. Secondly we introduce the principle of maximal aging which we term Einstein's First Law or the law of freefall. Thirdly we present an apparent conundrum, which is that non-falling would normally appear to be a shorter path in spacetime than a falling trajectory. Recognising this conundrum, we calculate the warping of the time axis required to make the free fall path shorter than the non-falling path. Simple approximations allow gravitational time dilation to be calculated by 14-15 year olds. Our result is approximate, and is a lower limit, but has the correct functional form.

Our Einsteinian gravity program is completed with discussion of the classic Gravity Probe A experiment that measured gravitational time dilation. We then go on to discuss atomic clocks and black holes, and how clocks will soon be able to become tools for geodesy. This also allows us present a relativistic formulation of gravitational potential energy that connects time dilation directly to $E=mc^2$: potential energy of mass m at any height h is equal to mc^2 times the fractional time dilation.

In section 2 we present our approach to Einsteinian gravity. In section 3 we briefly review research results that demonstrate that students both appreciate and comprehend the material presented.

II.

a) Introducing spacetime

i) Step 1: Units for measuring space and time.

The idea of four dimensional spacetime was introduced to the popular imagination by HG Wells in his novel The Time Machine in 1995. In the same year the idea was was suggested by an unknown author “S” in Nature magazine and in 1907 Minkowski presented the formulation of spacetime that is now a standard part of special relativity [(2)] While the idea of spacetime is central to the understanding of gravity, the idea does not gain clarity without discussion of units.

We measure time in seconds, an arbitrary unit probably derived from the human heart beat. Likewise we measure distances in units such as the foot (typical size of a human foot), yards (roughly a human pace) or in the modern era, meters, another arbitrary unit, which was supposed to be 1/40,000th of the Earth’s circumference. If we make a 2 dimensional spacetime diagram, - a graph of distance travelled in I dimension against time, the “length” of the spacetime trajectory is meaningless unless we can find a universal connection between space and time.

ii) Step 2: Finding the connection between space and time

The connection between space and time is familiar to young people because the connection is implicit in statements such as “it is five minutes up the road”. In the Einstein-First program we use roadside advertisements indicating distance to MacDonald’s to illustrate this idea. If Macdonald’s is five minutes away, what is the assumed speed. Students agree that on a freeway they must assume the freeway speed limit. On a footpath the speed assumed is walking speed.

From this emerges the idea that *speed* connects space and time and that spacetime only makes sense if we can agree about a universal speed to link space and time.

In the Einstein-First program we first ask students to plot 2 dimensional spacetime diagrams for simple journeys such as their journey to school. We use distance for the y-axis and time for the x-axis. Students quickly recognise that slope (gradient) represents speed, and that vertical paths are impossible. They interpret spacetime diagrams in terms of stationary periods, acceleration, and maximum possible speed.

iii) Step 3: Finding the universal connection between space and time.

The next key step is to understand that there is a universal connection between space and time. Depending on the age of the students we discuss historical studies including the Michelson-Morley experiment that proved the universality and constancy of the speed of light. Students quickly grasp the fact that the speed of light provides the universal connection between space and time.

With the speed of light recognised as a universal connection, we then consider the spacetime diagram for a falling apple. On Earth an apple falls 4.9 meters in one second. In spacetime units we measure time in meters: the apple falls 4.9 meters in 300 million metres of time. The spacetime diagram is highly elongated as shown in Figure 1. The y axis is 4.9 meters high but the x-axis is 3×10^8 meters long…stretching almost to the moon! This gives us a simple rule for graphing spacetime. If we measure distance in meters, we must also measure time in meters of light travel. An alternative (that we will not use here) is to express time in seconds, and then distances must be expressed in time for light to travel that distance. In this case we would say that 1 meter = 3 nanoseconds, or 3×10^{-9} seconds. In both cases a spacetime graph for the fall of normal objects near the earth is extremely elongated.

b) Principle of Maximal Aging: Einstein's First Law

The fundamental principle of general relativity is that freely falling trajectories are geodesics, the shortest paths in spacetime. In flat space a geodesic is a straight line: the shortest path between two points. Einstein's First Law is so-named because it is an exact 4D analogy to Newton's First Law, which states that bodies travel in straight lines unless acted on by an external force. Straight lines now refer to geodesics in curved spacetime. Einstein's First Law states that freely falling bodies follow geodesics in spacetime. Technically this means that the spacetime interval is minimal. This means that free fall brings us to our future destination (which is old age) in the shortest possible spacetime interval. Free fall is the fastest way to get old.

c) Which is the shortest trajectory: discovering time dilation.

The spacetime interval for a 2-dimensional spacetime diagram is simply the physical length of the spacetime trajectory that you could measure on a graph by following the line with a flexible tape measure. We will now see that Einstein's First law can only be true if the time axis is warped.

We consider the spacetime diagram for two possibilities: the trajectory of an apple which does not fall, and the trajectory of an apple which falls.

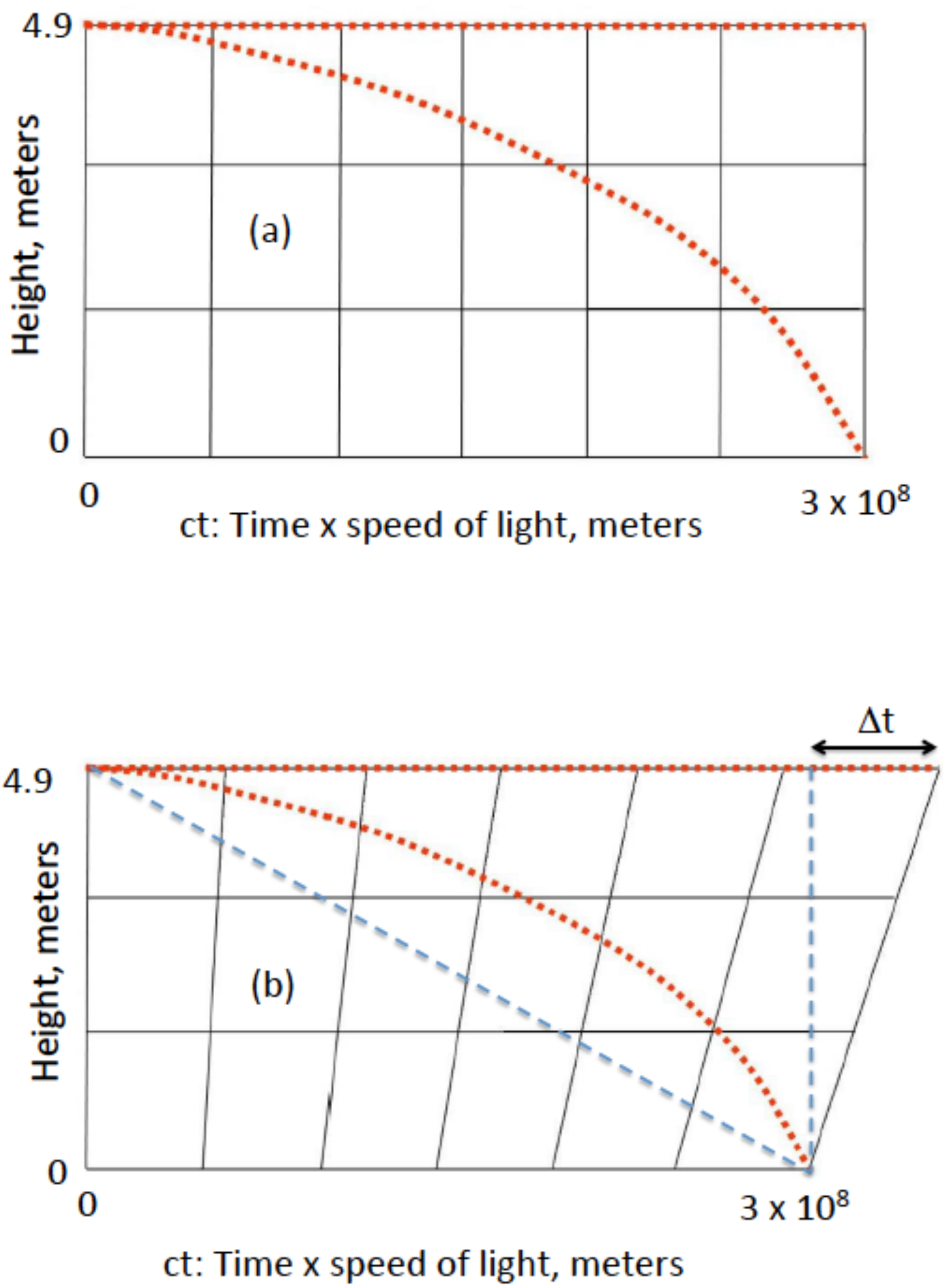

Figure 1: (a) Spacetime diagram for a falling apple and a non-falling apple in Euclidean coordinates. The apple falls 4.9 meters in 1 second, which in distance units is 3 x 10^8 meters. (b) Spacetime diagram for the same two possibilities, in which the time axis is stretched according to height. If Δt is large enough the non-falling trajectory can be longer than the falling trajectory. The diagonal line shows the construction used for estimating the magnitude of Δt.

The trajectory of the falling apple is a parabolic path on the spacetime diagram as shown in Figure 1, starting at 4.9m, and dropping to zero. The non-falling trajectory is a horizontal line at height = 4.9 m. On this graph the falling path is clearly longer than the non-falling path. But if Einstein's First Law is correct, then the non-falling path must be longest. There is only one way that this is possible: for the geometry to be non-Euclidean….for the time axis to be stretched as a function of height.

We can approximate all of this using Euclidean geometry and Pythagoras's theorem. Our aim is to discover the minimum stretching of the time axis required to make the non-falling trajectory longer than the free falling trajectory. Figure 2 shows the geometry for our approximation. The result we obtain will be a lower limit: Einstein's First Law says that free fall is the *shortest* spacetime interval. We will find the time dilation required for the interval to be equal.

First we will treat the spacetime interval for falling as the hypotenuse of a right angled triangle. Remembering that we must multiply time by c to express times in distance units, for height h we can then write

$$Interval^2 = h^2 + c^2t^2$$

We will now assume that the time axis at height h has been stretched by a small amount Δt, sufficient that the non-falling interval t+Δt is at least equal to the approximate free fall interval. Setting *interval* equal to $(t + \Delta t)$, we have

$$h^2 + c^2t^2 = (ct + c\Delta t)^2$$

We want to solve this equation for Δt/t, the fractional time dilation. It is very easy to simplify if we use the well known formula from Newtonian mechanics $s=1/2gt^2$, also remembering that because Δt is very small, Δt^2 is negligible. The equation can be re-written

$$h.\frac{1}{2}gt^2 + c^2t^2 = c^2t^2 + 2c^2t\Delta t + c^2\Delta t^2$$

Cancelling like terms, dividing by c^2t^2 and ignoring the term in Δt^2, the above equation becomes

$$\frac{\Delta t}{t} = \frac{gh}{4c^2}$$

The above lower limit for time dilation is ¼ of the actual general relativistic result $\Delta t/t = g/c^2$ per meter. The difference comes from the approximation of the parabola as a triangle, and because time dilation causes the free falling path to be shorter, and not equal to the non-free-fall path.

III. Significance of gravity as time dilation

We have seen that Einstein's First Law can only hold if there is gravitational time dilation. Clocks must run faster at high altitude compared with the ground, by an amount g/c^2 per meter, or one part in 10^{16} per meter. This tells us that time travels faster by about 30 microseconds per year on top of Mt Everest, and approaching 1 millisecond per year in low Earth orbit. Equally it tells us that the Earth is a time machine. The forces exerted on us by the Earth, that cause us to deviate from a natural free fall trajectory, slow the passage of time and hence cause aging to be reduced by significant fraction of a second in a lifetime.

The Wallal eclipse expedition of 1922 proved the spatial curvature due to the sun, to a precision ~ 1%, [(3)] but the effect of gravity on time was unproven until two classic experiments: the 1959 Rebka and Pound experiment [(4)] that measured gravitational time dilation for x-ray photons and Vessot's Gravity Probe A experiment[(5)] that measured the same effect using an atomic clock launch to high altitude. These two experiments showed that time dilation acts identically for a freely moving photon as for a clock which counts the oscillations of an atomic system.

The approximate derivation presented here, based only on the assertion that objects in free fall follow geodesics in space time, show that deep conclusions from general relativity can be plausibly derived without complex mathematics, at a level understandable by high school students.

Gravitational potential energy. Another interesting observation is that the familiar concept of gravitational potential energy can be linked to general and special relativity if you make a trivial manipulation. From $\Delta t/t = gh/c^2$, it follows immediately that

$$mgh = \frac{\Delta t}{t}.mc^2$$

Gravitational potential energy is the rest mass energy times the fractional time dilation. This interesting result again implies the close link between gravity and time dilation. Gravity is a direct manifestation of time dilation.

Students are very interested in black holes. The results presented here for uniform gravity, can be expressed very simply in a more astronomical context, by measuring the distance R from the centre of a planet in units of the planet's gravitational radius, where this radius is simply the Schwartzchild radius it would have if were to be converted to a black hole. That is $\Delta t/t = R_s/R = 2GM/c^2R$. At the Earth's surface $\Delta t/t = 9 \times 10^{-3}/6 \times 10^6 = 1.5 \times 10^{-9}$ or ~ 50ms per year.

Newton's Law of Gravity. Is Newton's Law of Gravitation described by the formula $F=GM_1M_2/r^2$ still correct? The answer is that the inverse square law appears to be valid for static gravitational forces measured in the laboratory. Searches over many decades have failed to find any violations of this law. However the law fails to correctly describe dynamical phenomena such as orbits, because of two effects: the curved space effects that cause the laws of Euclidean geometry to fail, and gravitomagnetism or frame dragging in which rotation of a massive body distorts the mass distribution seen by a nearby body, changing the motion of nearby objects. In addition, energy associated with the curved spacetime near a black hole causes the measured mass of a black hole to be dependent on distance from the black hole. At an infinite distance the mass of the black hole appears to be double the mass you would measure close up (for example by measuring orbits of a test mass). Thus Newton's Law of Gravitation would seriously fail if you measured the force between two black holes as a function of distance. Thus we can say definitively that the Law of Gravitation has been overthrown and replaced with the new concepts of curved spacetime. It is clearly important to emphasise that Newtonian gravity it is an extremely useful approximation, a useful tool, and valid to high precision in the solar system.

Students who use GPS navigators in their phones need to know that accurate understanding of gravitational time dilation is critical to their operation and that either hardware or software corrections must be applied to the time signals from GPS satellites to obtain accurate position information.

Why we should teach Einsteinian Gravity. In spite of modern science, the general belief among educators is that Einstein's physics is too difficult to teach in school. As a result, science students enter university indoctrinated with 2,300-year-old Euclidean geometry and 300-year-old Newtonian physics. Very few go on to discover the Einsteinian reality of curved space, warped time and quantum weirdness. The lucky few who get to study Einsteinian physics have difficulties because the fundamental concepts contradict their past learning.

Most students who go on to become school teachers maintain the Newtonian mindset and so education remains in a Euclidean timewarp! The drastic decline in science at school and university could in part be due to our failure to challenge young people with modern ideas such as these.

If we start young enough, and develop adequate materials, our studies with 11-15 year olds suggest that young people can easily learn that the world is non-Euclidean, and then appreciate that the geometrical formulae we learn at school, and Newton's Law of Gravitation, are convenient approximations for everyday life.

In future papers we will describe other aspects of our Einstein-First Program(6,7) that endeavours to make Einsteinian physics, non-Euclidean geometry and quantum mechanics the first contact with physics at school.

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